International Journal of Theoretical Physics, Vol. 39, No. 4, 2000

Exact Solutions of the Two-Dimensional Schrödinger Equation with Certain Central Potentials

Shi-Hai Dong[1]

Received September 29, 1999

By applying an ansatz to the eigenfunction, an exact closed-form solution of the Schrödinger equation in two dimension is obtained with the potentials $V(r) = ar^2 + br^4 + cr^6$, $V(r) = ar + br^2 + cr^{-1}$, and $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$, respectively. The restrictions on the parameters of the given potential and the angular momentum m are obtained.

1. INTRODUCTION

One of the important tasks of quantum mechanics is to solve the Schrödinger equation with physical potentials. It is well known that the exact solution of the Schrödinger equation is possible only for certain potentials such as Coulomb or harmonic oscillator potentials. Approximation methods are frequently used to obtain the solution. In the past several decades, much effort have been made to study the stationary Schrödinger equation with central potentials containing negative powers of the radial coordinate [1–31]. Generally, most of these authors treated these problems in three-dimensional space. Recently, the study of higher order central potentials has been of interest to physicists and mathematicians who want to understand newly discovered physical phenomena such as structural phase transitions [1], polaron formation in solids [2], and the concept of false vacuo in field theory [3]. In addition the solution of the Schrödinger equation with the sextic potential $V(r) = ar^2 + br^4 + cr^6$ can be applied in the field of fiber optics [4], where one wants to solve a similar problem of an inhomogeneous spherical or circular

[1] Physical and Theoretical Chemistry Laboratory, University of Oxford, Oxford OX1 3QZ, U.K., and the Department of Physics, Cardwell Hall, Kansas State University, Manhattan, Kansas 66506; e-mail: dongsh@phys.ksu.edu.

0020-7748/00/0400-1119$18.00/0

waveguide with refractive index profile a function of a sextic-type potential. Its solution is also applicable to molecular physics [5]. The study of the mixed potential $V(r) = a_1 r + b_1 r^2 + c_1 r^{-1}$ (harmonic + linear + Coulomb) as a phenomenological potential appears in nuclear physics. The study of the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$ has been widely used in different fields such as atomic physics and optical physics [29–31]. Interest in these anharmonic oscillator-like interactions stems from the fact that the study of the relevant Schrödinger equation, for example, in atomic and molecular physics as well as nuclear physics, provides us with insight into the physical problem in question.

With the wide interest in the lower dimensional field theory in the recent literature, however, it is necessary to study the two-dimensional Schrödinger equation with certain central potentials such as the sextic and mixed potentials as well as the singular even-power potential, an investigation which, to our knowledge, has not appeared in the literature. Furthermore, two-dimensional models are often applied to make the more involved higher dimensional systems tractable. Therefore, it seems reasonable to study the two-dimensional Schrödinger equation with these potentials, which is the purpose of this paper. On the other hand, we have succeeded in studying the two-dimensional Schrödinger equation with some anharmonic potentials [16, 17].

This paper is organized as follows. Section 2 studies the solution of the two-dimensional Schrödinger equation with the sextic potential $V(r) = ar^2 + br^4 + cr^6$ using an ansatz for the eigenfunction. The study of the mixed potential $V(r) = a_1 r + b_1 r^2 + c_1 r^{-1}$ is presented in Section 3. In Section 4, we will study the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$. A brief conclusion is given in Section 5.

2. THE SEXTIC POTENTIAL

Throughout this paper the natural units $\hbar = 1$ and $\mu = 1/2$ are employed. Consider the two-dimensional Schrödinger equation with a potential $V(r)$ that depends only on the distance r from the origin,

$$H\psi = -\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\psi + V(r)\psi = E\psi \tag{1}$$

where the potential is taken as

$$V(r) = ar^2 + br^4 + cr^6 \tag{2}$$

The choice of r, φ coordinates reflects a model where the full Hilbert space is the tensor product of the space of square-integrable functions on the positive half-line with the space of square-integrable functions on the circle. We therefore write

$$\psi(\mathbf{r}, \varphi) = r^{-1/2} R_m(r) e^{\pm im\varphi}, \qquad m = 0, 1, 2, \ldots \tag{3}$$

and this factorization leads to a second-order equation for the radial function $R_m(r)$ with vanishing coefficient of the first derivative, i.e.,

$$\frac{d^2 R_m(r)}{dr^2} + \left[E - V(r) - \frac{m^2 - 1/4}{r^2}\right] R_m(r) = 0 \tag{4}$$

where m and E denote the angular momentum and energy, respectively. For the solution of Eq. (4), we make an ansatz [6–12] for the radial wave function

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{2n+\delta} \tag{5}$$

where

$$p_m(r) = \frac{1}{2}\alpha r^2 + \frac{1}{4}\beta r^4 \tag{6}$$

Substituting Eq. (5) into Eq. (4) and equating the coefficient of $r^{2n+\delta+2}$ to zero, we obtain

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{7}$$

where

$$A_n = \alpha^2 + (3 + 2\delta + 4n)\beta - a \tag{8a}$$

$$B_n = E + (1 + 2\delta + 4n)\alpha \tag{8b}$$

$$C_n = (\delta + 2n)(-1 + \delta + 2n) - (m^2 - 1/4) \tag{8c}$$

and

$$\beta^2 = c \tag{9a}$$

$$2\alpha\beta = b \tag{9b}$$

It is easy to obtain the values of parameters for $p_m(r)$ from the Eq. (9) written as

$$\beta = \pm\sqrt{c}, \qquad \alpha = \frac{b}{2\beta} \tag{10}$$

If the first nonvanishing coefficient $a_0 \neq 0$ in Eq. (7), we can obtain $C_0 = 0$ from Eq. (8c), i.e., $\delta = -m + 1/2$ or $m + 1/2$. In order to retain the well-behaved solution at the origin and at infinity, we choose δ and β as follows:

$$\delta = m + 1/2, \qquad \beta = -\sqrt{c} \tag{11a}$$

from which one can obtain

$$\alpha = -\frac{b}{2\sqrt{c}} \tag{11b}$$

On the other hand, if the pth nonvanishing coefficient $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \cdots = 0$, it is easy to obtain $A_p = 0$ from Eq. (8a), i.e.,

$$a + 2\sqrt{c}(2 + m + 2p) - \frac{b^2}{4c} = 0 \tag{12}$$

which is a restriction on the parameters a, b, c of the potential and angular momentum m and p ($p \leq n$). As we know, A_n, B_n, and C_n must satisfy the determinant relation for a nontrivial solution

$$\det\begin{vmatrix} B_0 & C_1 & \cdots & \cdots & \cdots & 0 \\ A_0 & B_1 & C_2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{p-1} & B_p \end{vmatrix} = 0 \tag{13}$$

In order to present this method, we will give the exact solutions for the different $p = 1, 2$ as follows.

1. When $p = 0$, it is easy to obtain $B_0 = 0$ from Eq. (13), which, together with Eq. (11), leads to

$$E_0 = \frac{3b}{2\sqrt{c}} \tag{14}$$

In this case, however, the restriction on the parameters of the potential and the angular momentum m is obtained as

$$a + 2\sqrt{c}(2 + m) - \frac{b^2}{4c} = 0 \tag{15}$$

The corresponding eigenfunction for $p = 0$ can now be written as

$$R_m^{(0)} = a_0 r^{\delta} \exp\left[-\frac{b}{4\sqrt{c}}\, r^2 - \frac{\sqrt{c}}{4}\, r^4\right] \tag{16}$$

where a_0 is the normalization constant and δ is given by Eq. (11).

2. When $p = 1$, one can arrive at the following relation from Eq. (13):

$$B_0 B_1 - A_0 C_1 = 0 \tag{17}$$

Similarly, we can obtain the energy eigenvalue from Eqs. (7)–(10) as

$$E_1 = \frac{b(2 + m)}{\sqrt{c}} \pm \frac{\sqrt{b^2(2 + m) - 4c(1 + m)(2 + 2\sqrt{c}(2 + m))}}{\sqrt{c}} \tag{18}$$

The corresponding restriction on the parameters and m is

$$a + 2(4 + m)\sqrt{c} - \frac{b^2}{4c} = 0 \tag{19}$$

The corresponding eigenfunction for $p = 1$ is

$$R_m^{(1)} = (a_0 + a_1 r^2) r^{\delta} \exp\left(-\frac{b}{4\sqrt{c}}\, r^2 - \frac{\sqrt{c}}{4}\, r^4\right) \tag{20}$$

where δ has been given by Eq. (11), and the coefficients a_0 and a_1 can be determined completely by the normalization condition

In this way, we can generate a class of exact solutions by setting $p = 1, 2, \ldots$. For the general case, if the pth nonvanishing coefficient $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, we can obtain $A_p = 0$, i.e.,

$$\alpha^2 + (3 + 2\delta + 4p) = a \tag{21}$$

The corresponding eigenfunction is

$$R_m^{(p)} = (a_0 + a_1 r^2 + \cdots + a_p r^{2p}) r^{\delta} \exp\left[-\frac{b}{4\sqrt{c}}\, r^2 - \frac{\sqrt{c}}{4}\, r^4\right] \tag{22}$$

where δ has been given by Eq. (11a), and a_i ($i = 1, 2, \ldots, p$) can be expressed by the recurrence relation (7) and in principle obtained by the normalization condition.

3. THE MIXED POTENTIAL

The approach for this potential is similar to that for the sextic potential except for taking the ansatz as

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{n+\delta} \tag{23}$$

where p_m is taken as

$$p_m(r) = \alpha r + \frac{1}{2}\beta r^2 \tag{24}$$

We can solve the two-dimensional Schrödinger equation with the potential

$$V(r) = ar + br^2 + \frac{c}{r} \tag{25}$$

Similarly, we obtain the following sets of equations after substituting Eq. (23) into Eq. (4) and equating the coefficients of $r^{\delta+n}$ to zero:

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{26}$$

where

$$A_n = E + \beta(1 + 2n + 2\delta) \tag{27a}$$

$$B_n = -c + \alpha(2n + 2\delta) \tag{27b}$$

$$C_n = (n + \delta)(-1 + n + \delta) - (m^2 - 1/4) \tag{27c}$$

and

$$\beta^2 = b, \qquad 2\alpha\beta = a \tag{27d}$$

Similar to the above choices, we can choose β and δ as $-\sqrt{b}$ and $m + 1/2$, respectively. According to these choices, the parameter α can be obtained as

$$\alpha = -\frac{a}{2\sqrt{b}} \tag{28}$$

Now, let us consider the case $a_p \neq 0$, but $a_{p+1} = a_{p+1} = \cdots = 0$; then we can get $A_p = 0$. In this case, the energy eigenvalue can be written as

$$E_p = 2\sqrt{b}(1 + m + p) \tag{29}$$

Likewise, the nontrivial solution of the recursion relation (26) can be obtained by Eq. (13). The exact solutions for $p = 0$ and $p = 1$ are discussed below.

1. When $p = 0$, we arrive at

$$E_0 = 2\sqrt{b}(1 + m) \tag{30}$$

and $B_0 = 0$, i.e.,

$$2c\sqrt{b} = a(1 + 2m) \tag{31}$$

which is a restriction on the corresponding parameters of the potential and the angular momentum m. The eigenfunction can be given as

$$R_m^{(0)} = a_0 r^{\delta} \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right] \tag{32}$$

where δ is taken as $m + 1/2$, and the coefficient a_0 can be evaluated by the normalization condition.

2. When $p = 1$, the energy eigenvalue can be written as

$$E_1 = 2\sqrt{b}(2 + m) \tag{33}$$

Moreover, we can obtain the restriction on the parameters of the potential and the angular momentum m from the determinant relation (13) as $B_0 B_1 = A_0 C_1$, i.e.,

$$\left\{c + \frac{(1 + 2m)a}{2\sqrt{b}}\right\}\left\{c + \frac{(3 + 2m)a}{2\sqrt{b}}\right\} = 2\sqrt{b}(1 + 2m) \tag{34}$$

In this case, the corresponding eigenfunction can be written as

$$R_m^{(1)} = (a_0 + a_1 r)r^{\delta} \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right] \tag{35}$$

where a_0 and a_1 can be obtained by the recursion relation (26) and the normalization relation.

Similarly, if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, we get $A_p = 0$. In this case, the eigenfunction can be written as

$$R_m^{(p)} = (a_0 + a_1 r + \cdots + a_p r^p)r^{\delta} \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right] \tag{36}$$

where δ is taken as $m + 1/2$, and the coefficients $a_i (i = 1, 2, \ldots, p)$ can be calculated by Eq. (26) and the normalization condition.

4. THE SINGULAR EVEN-POWER POTENTIAL

Similar to the above discussion, for the central singular even-power potential we can take the ansatz

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{2n+\delta} \tag{37}$$

where p_m is taken as

$$p_m(r) = \frac{1}{2}\alpha r^2 + \frac{1}{2}\beta r^{-2} \tag{38}$$

We can solve the two-dimensional Schrödinger equation with the potential

$$V(r) = ar^2 + \frac{b}{r^2} + \frac{c}{r^4} + \frac{d}{r^4} \tag{39}$$

We can get the following sets of equations after substituting the ansatz (37) into Eq. (4) and equating the coefficients of $r^{\delta+n}$ to zero:

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{40}$$

where

$$A_n = E + \alpha(1 + 2\delta + 4n) \tag{41a}$$

$$B_n = -b - 2\alpha\beta - (m^2 - 1/4) + (\delta + 2n)(-1 + \delta + 2n) \tag{41b}$$

$$C_n = (3 - 2\delta - 4n) - c \tag{41c}$$

and

$$\alpha^2 = a, \qquad \beta^2 = d \tag{42}$$

Similar to the above choices, we can choose α and β as $-\sqrt{a}$ and $-\sqrt{d}$, respectively. Moreover, if $a_0 \neq 0$, then one can obtain $C_0 = 0$, i.e.,

$$\delta = (3/2 + \mu) \tag{43}$$

where $\mu \equiv c/(2\sqrt{d})$. However, if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, then $A_p = 0$, from which one can obtain the energy eigenvalue as

$$E_p = \sqrt{a}(4 + 4p + 2\mu) \tag{44}$$

We now discuss the corresponding exact solutions for $p = 0$ and $p = 1$.

1. When $p = 0$, we arrive at

$$E_0 = \sqrt{a}(4 + 2\mu) \tag{45}$$

In this case, $B_0 = 0$ from the determinant relation (13), which leads to the constraint condition between the parameters of the potential and the angular momentum quantum number m,

$$(1 + \mu)^2 - b - 2\sqrt{ad} - m^2 = 0 \tag{46}$$

The corresponding eigenfunction/can be written as

$$R_m^{(0)} = a_0 r^{\delta} \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right] \tag{47}$$

where δ is given by Eq. (43) and the coefficient a_0 can be obtained by the normalization condition.

2. When $p = 1$, the energy eigenvalue can be obtained from Eq. (44) as follows:

$$E_1 = \sqrt{a}(8 + 2\mu) \tag{48}$$

In this case, the determinant relation (13) gives $B_0B_1 = A_0C_1$, which results in the following restriction on the parameters and angular momentum quantum m:

$$\begin{aligned}&[-b - 2\sqrt{ad} + (1 + \mu)^2 - m^2]\\ &\quad\times [-b - 2\sqrt{ad} + (3 + \mu)^2 - m^2] - 16\sqrt{ad} = 0\end{aligned} \tag{49}$$

The eigenfunction for $p = 1$ is

$$R_m^{(1)} = (a_0 + a_1 r^2) r^\delta \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right] \tag{50}$$

where δ is given by Eq. (43), and a_i ($i = 0, 1$) can be calculated from Eq. (40) and the normalization relation. Following this method, we can obtain a class of exact solutions by setting the different p. Generally, the corresponding eigenfunction for p can be written as

$$R_m^{(p)} = (a_0 + a_1 r^2 + \cdots + a_p r^{2p}) r^\delta \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right] \tag{51}$$

where a_i ($i = 0, 1, \ldots, p$) can be evaluated from recursion relation (40) and the normalization condition.

5. CONCLUDING REMARKS

In this paper, applying an ansatz to the eigenfunction, we have obtained the exact solutions of the two-dimensional Schrödinger equation with the sextic potential $V(r) = ar^2 + br^4 + cr^6$, the mixed potential $V(r) = ar + br^2 + cr^{-1}$ as well as the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$. The corresponding restrictions on the parameters of the potential and the angular momentum m have been obtained for the different potentials. The study of other classes of central potentials by this method is in progress.

ACKNOWLEDGMENTS

The author gratefully acknowledges Prof. Zhong-Qi Ma for encouragement and Prof. Mark S. Child for reading the manuscript carefully. I am also grateful to Dr. Xiao-Gang Wang for help at Oxford. This work was supported in part by the Royal Society, and supported in part by the Division of Chemical Sciences, Office of Basic Energy Sciences, Office of Energy Research, US Department of Energy.

REFERENCES

1. A. Share and S. N. Behra, *Pramana J. Phys.* **14** (1980).
2. D. Amin, *Phys. Today* **35**, 35(1982); *Phys. Rev. Lett.* **36**, 323 (1976).
3. S. Coleman, *Aspects of Symmetry,* (Cambridge University Press, Cambridge, 1988), p. 234.
4. H. Hashimoto, *Int. J. Electron.* **46**, 125 (1979); *Opt. Commun.* **32**, 383 (1980).
5. C. E. Reid, *J. Mol. Spectrosc.* **36**, 183 (1970).
6. R. S. Kaushal, *Ann. Phys.* (N.Y.) **206**, 90 (1991).
7. R. S. Kaushal and D. Parashar, *Phys. Lett. A* **170**, 335 (1992).

8. R. S. Kaushal, *Phys. Lett. A* **142**, 57 (1989).
9. S. K. Bose and N. Varma, *Phys. Lett. A* **147**, 85 (1990).
10. S. K. Bose, *Nuovo Cimento B* **109**, 1217 (1994).
11. A. Voros, *J. Phys. A* **32**, 5993 (1999).
12. Y. P. Varshni, *Phys. Lett. A* **183**, 9 (1993).
13. S. Özcelik and M. Simsek, *Phys. Lett. A* **152**, 145 (1991).
14. M. Simsek and S. Özcelik, *Phys. Lett. A* **186**, 35 (1994).
15. M. Simsek, *Phys. Lett. A* **259**, 215 (1999).
16. Shi-Hai Dong and Zhong-Qi Ma, *J. Phys. A* **31**, 9855 (1998).
17. Shi-Hai Dong, Zhong-Qi Ma, and G. Esposito, *Found. Phys. Lett.* **12**, 465 (1999).
18. M. Znojil, *J. Math. Phys.* **30**, 23 (1989).
19. M. Znojil, *J. Math. Phys.* **31**, 108 (1990).
20. M. Znojil, *J. Phys. A* **15**, 2111 (1982).
21. V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965).
22. S. Fubini and R. Stroffolini, *Nuovo Cimento* **37**, 1812 (1965).
23. F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic, Press, New York, 1967).
24. R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1967).
25. W. M. Frank, D. J. Land, and R. M. Spector, *Rev. Mod. Phys.* **43**, 36 (1971).
26. R. Stroffolini, *Nuovo Cimento A* **2**, 793 (1991).
27. G. Esposito, *J. Phys. A* **31**, 9493 (1998).
28. G. Esposito, *Found. Phys. Lett.* **11**, 535 (1998).
29. A. O. Barut, *J. Math. Phys.* **21**, 568 (1980).
30. B. H. Bransden and C. J. Joachain, *Physics of Atoms and Molecules* (Longman, London, 1983).
31. G. C. Maitland, M. M. Higby, E. B. Smith, and V. A. Wakoham, *Intermolecular Forces* (Oxford University Press, Oxford, 1987).